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Gleason-Type Theorems for Signed Measures on Orthomodular Posets of Projections on Linear Spaces

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We consider some orthomodular posets which are not lattices and the Gleason-type theorems for signed measures on them.

The basic structure of the quantum probability theory is the set $\Pi(H)$ of all Hermitian projections on a complex or real Hilbert space H. This set is endowed with the orthomodular partially ordered set (OMPoset) structure, i.e., with an ordering $\leq$ ($P \leq Q$ iff $PQ = QP = P$) and with an involutive antiautomorphism $(\cdot)'$ ($P' = I - P$) which satisfy the axioms:

1. there exist the greatest, 1, and the least, 0, elements.
2. $a \leq b$ implies $b' \leq a'$, $(a')' = a$, $0' = 1$.
3. If $a \leq b'$ (in this case we write $a \perp b$ and call a and b *orthogonal*), then there exists the supremum $a \vee b$.
4. If $a \leq b$, then there exists a $c \leq a'$ such that $b = a \vee c$.

Studies have been made of probability measures (states) on $\Pi(H)$, i.e., maps μ: $\Pi(H) \to [0, 1]$ such that:

(S1) $\mu(I) = 1$.

(S2) $\mu(P_1 + P_2 + \cdots) = \mu(P_1) + \mu(P_2) + \cdots)$ for every sequence of mutually orthogonal projections in $\Pi(H)$.

The well-known Gleason (1957) theorem establishes a representation for every state μ on $\Pi(H)$ in the case $\dim(H) \geq 3$:

$$\mu(P) = \mathrm{tr}(TP) \qquad \text{for all} \quad P \in \Pi(H), \quad T \text{ a nuclear positive operator} \qquad (1)$$

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In Mushtari (1989) I proposed to develop a measure theory for another OMP of projections, namely the set $\mathfrak{P}(H)$ of all (not necessarily Hermitian) continuous projections on H. This set is endowed with the OMP structure in the same way as $\Pi(H)$ is. In Section 1, I draw the attention of the reader to some results of Mushtari (1989) concerning the OMP $\mathfrak{P}(H)$, in particular, to an analog for the Gleason theorem. In Section 2, a new definition of a signed measure on $\mathfrak{P}(H)$ is given which is defined with the OMP structure. In Section 3, I discuss some possible generalizations of such Gleason-type theorems for OMP $\mathfrak{P}(X)$ of projections on a non-Hilbertian topological linear (or simply linear) space X over some field F. I sketch a proof of the first result of this kind, namely for the case when X is a finite-dimensional linear space over the field $\mathbf{Q}$ of rationals.

1. THE OMP $\mathfrak{P}(H)$

Now I give a general idea of the transition $\Pi(H) \to \mathfrak{P}(H)$.

Definition 1. An ordered pair (a, b) of elements of a lattice T is called (Maeda and Maeda, 1970) an M-pair, and we write $(a, b)M$ [respectively an M^*-pair and we write $(a, b)M^*$], if

$$a \wedge (b \vee c) = (a \wedge b) \vee c \qquad \text{for all} \quad c \leq a \tag{2}$$

[respectively $(a \vee b) \wedge c = a \vee (b \wedge c)$ for all $c \geq a$].

Definition 2. A lattice T is called *M-symmetric* (respectively *M^*-symmetric*) if $(a, b)M$ implies $(b, a)M$ [respectively $(a, b)M^*$ implies $(b, a)M^*$].

Theorem 1. Let T be an M-symmetric and M^*-symmetric lattice with the greatest, 1, and the least, 0, elements and let L be the set of all (a, b) in T such that (a) $a \vee b = 1$, (b) $a \wedge b = 0$, (c) $(a, b)M$, $(a, b)M^*$. Then L is an OMP endowed with (i) the order relation $(a, b) \leq (c, d)$ iff $a \leq c$, $b \geq d$, and (ii) the involutive antiautomorphism $(a, b)' = (b, a)$.

Theorem 2. Let X be a Hausdorff linear topological space and E and F be elements of the lattice $\Pi(X)$ of all closed linear subspaces of X. A pair (E, F) is an M^*-pair if and only if $E + F$ is closed.

Corollary. The lattice $\Pi(X)$ of all closed linear subspaces of a Hausdorff locally convex space X is M-symmetric and M^*-symmetric. The OMP $\mathfrak{P}(H)$ is constructed from $\Pi(X)$ as in Theorem 1.

Remark 1. The simplest five-element nonmodular lattice is not M-symmetric and not M^*-symmetric.

Now we proceed to study the case when X coincides with a real Hilbert space H. Unfortunately, the OMP $\mathfrak{P}(H)$ [as well as the measure theory on

$\mathfrak{P}(H)$] does not have some good properties of $\Pi(HY)$. It is easy to see that the following statements are true.

Proposition 1. (i) If $\dim(H) > 2$, then $\mathfrak{P}(H)$ is not a lattice. (ii) If $\dim(H) = \infty$, then $\mathfrak{P}(H)$ is not a σ-orthocomplete OMP [i.e., there exists a sequence of mutually orthogonal projections $P_n \in \mathfrak{P}(H)$ such that the supremum $\vee_n P_n$ does not exist]. (iii) $\mathfrak{P}(H) = \cup\{\Pi_A$: A runs over the set of all positive invertible operators on $H\}$, where Π_A is the OMP of all projections which are Hermitian with respect to the scalar product $(x, y)_A = (Ax, y)$.

Proposition 2. If μ is a nonzero function on $\mathfrak{P}(H)$ defined by (1) with a nuclear operator T and if $\mu \geq 0$ on $\mathfrak{P}(H)$, then $\dim(H) < \infty$ and $T = \mathrm{const} \cdot I$. If $\dim(H) = \infty$, then any nonzero signed measure μ on $\mathfrak{P}(H)$ is not σ-additive.

So, if we replace one metric $(\cdot, \cdot)$ by another metric $(\cdot, \cdot)_A$, we spoil a state and make it a signed measure. Therefore, in the case of the OMP $\mathfrak{P}(H)$ we have to deal with the signed measures, but no states. Unfortunately, even in the classical case of the OMP $\Pi(H)$ the Gleason theorem for signed measures μ is not true for finite-dimensional H. In fact, it is well known that there exists an additive $[f(x + y) = f(x) + f(y)]$ but not linear function f on the real line $\mathbf{R}$. Using this function, we can spoil a still good signed measure μ on $\Pi(H)$, defined by the relation (1). We replace μ by $f \circ \mu$. It is easy to see that $f \circ \mu$ is a signed measure and does not satisfy (1). So the Gleason theorem for finite-dimensional signed measures need some supplementary conditions, namely that of the boundedness of μ.

This difficulty leads to the following definition of signed measures on the OMP $\mathfrak{P}(H)$

Definition 3. A function μ: $\mathfrak{P}(H) \to \mathbf{R}$ is called a signed measure if its restriction to every sub-OMP $\Pi_A(H)$ is σ-additive and bounded.

Theorem 3. Let H be a real Hilbert space, $\dim(H) \geq 3$. Then every signed measure μ on the OMP $\mathfrak{P}(H)$ admits the representation (1) with a unique nuclear operator T.

Sketch of the Proof. It follows from the classical Gleason representation theorem that (1) holds for the restriction of μ to every $\Pi_A(H)$ with some operator $T = T(A)$. Our task is to prove that all the $T(A)$ can be chosen equal to each other. In order to establish this, we use the following obvious remark: for all different A and B the operators $T(A)$ and $T(B)$ need be such that the restrictions of the defined by (1) and $T(A)$ or by (1) and $T(B)$ signed measures to the intersection $\Pi_A(H) \cap \Pi_B(H)$ are equal to each other and to the restriction of μ. In order to simplify the calculation we represent the signed measure μ as a sum $\mu = \mu_H + \mu_S$ of the Hermitian part μ_H [i.e., $\mu_H(P) =$

$\mu_H(P^*)$ for all P] and the skew-Hermitian one μ_S [$\mu_S(P) = -\mu_S(P^*)$ for all P]. We prove (1) for μ_H and μ_S separately. In the case $\mu = \mu_H$ we establish the equality $T(A) = T(I)$ for all A where all $T(A)$ are chosen Hermitian. For this we need the Gleason representation only for the projections on the two-dimensional subspaces (taking into account that the whole H has a greater dimension and so the Gleason theorem holds). The case $\mu = \mu_S$, where T is chosen skew-Hermitian, is more difficult. In fact, $\Pi = \Pi(I)$ consists of Hermitian operators and $\mu_S = 0$ on Π, therefore the operator $T(I)$ may be chosen arbitrarily. So it is not easy to fix T. That is why in this case we use the three-dimensional Gleason theorem. The procedure above permits us to establish the theorem in the finite-dimensional case. In the infinite-dimensional case we have some supplementary difficulty in establishing the nuclearity of T.

An analogous theorem [a slightly modified (1)] is true for signed measures $\mathfrak{P}(H)$ in the case of the complex Hilbert space H.

Remark 2. Using a technique developed for the Hermitian case Matvejchuk (1981), this theorem is generalized to signed measures defined on $\mathfrak{P}$ for various classes of von Neumann algebras (Matvejchuk, 1991).

2. AN INTERNAL GLEASON THEOREM FOR $\mathfrak{P}(H)$

Now we will discuss the question: Is an internal formulation of Theorem 3 possible? Unfortunately, Definition 3 of a signed measure is not internal, because subOMPosets $\Pi_A(H)$ are defined from the metrical structure of the space H and not only from the ordering and the involutive antiautomorphism on the OMP $\mathfrak{P}(H)$. So, some internal reformulation of this definition should be of interest. Such a reformulation is possible in the infinite-dimensional case because of a nice theorem due to Dorofeev and Sherstnev (1990). Namely they have proved that in the case $\dim(H) = \infty$ every σ-additive signed measure on $\Pi(H)$ is bounded automatically. As we have seen in Proposition 2, we must give a new definition of σ-additivity which should be not only internal but also valid for the functions defined by (1). With this aim in view we introduce the following definition:

Definition 4. Let $\dim H = \infty$. A function μ: $\mathfrak{P}(H) \to \mathbf{R}$ is called a σ-additive signed measure if its restriction to every σ-subalgebra of $\mathfrak{P}(H)$ is σ-additive.

Theorem 4. Let $\dim(H) = \infty$. Then a function μ: $\mathfrak{P}(H) \to \mathbf{R}$ is a σ-additive signed measure if and only if it satisfies (1).

The part "only if" follows from Theorem 3 and the theorem of Dorofeev and Sherstnev. The part "if" follows from the following general property of sequenes in Banach reflexive spaces.

Theorem 5. Let (P_n) be a sequence of continuous pairwise orthogonal projections on a reflexive Banach space X; let it be given that any subsequence $P_{n(k)}$ of (P_n) has the supremum in $\mathfrak{P}(X)$. Then a series $\Sigma_n P_n$ is unconditionally pointwise convergent in the strong topology.

Obviously if the series $\Sigma_n P_n$ satisfies the condition in Theorem 5, then for every nuclear operator T the series $\Sigma_n \operatorname{tr}(TP_n)$ is convergent. So the new formulation of the σ-additive signed measure is not only internal, but also it implies the Gleason theorem.

3. GLEASON-TYPE THEOREM FOR PROJECTIONS ON RATIONAL LINEAR SPACES

The definition of the σ-additive signed measure we propose is not related to any metric or scalar product structure, it does not use the scalar products on the space X. Therefore, Definition 3 gives the possibility to formulate (unfortunately only formulate) some generalizations of Theorem 4 for various infinite-dimensional linear topological spaces, for example, for real or complex Banach and linear topological spaces and even for p-adic linear topological spaces. In these cases good scalar products on the space is not known.

But how should one prove such theorems? The classical Gleason theorem does not have sense without the OMP II defined by means of the scalar product structure and so cannot be used in these cases, unlike the proof of Theorem 3. In the general situation we have lost this basic tool. Obviously we may artificially introduce scalar products on some finite-dimensional subspaces, but this gives nothing, as the Gleason theorem in the finite-dimensional case is not valid. We see that any representation (1) being proved without use of the classical Gleason theorem for states would be of interest.

The rest of the paper presents an attempt to find the new tool we need. Let us summarize what we can do. What troubles us? The function f and the counterexample $f \circ \mu$ must vanish if we consider f only on $\mathbf{Q}$ and so μ taking values in $\mathbf{Q}$. Therefore, we come to the following theorem.

Theorem 6. Let $\mathfrak{P}(\mathbf{Q}^n)$ be the OMP of all linear projections on the space $\mathbf{Q}^n$ over the rationals field, $n \geq 3$, μ: $\mathfrak{P}(\mathbf{Q}^n) \rightarrow \mathbf{Q}$ be a signed measure, i.e.,

$$\mu(P_1 + P_2) = \mu(P_1) + \mu(P_2) \tag{3}$$

for mutually orthogonal projections P_1 and P_2 in $\mathfrak{P}(\mathbf{Q}^n)$. Then μ satisfies the Gleason representation.

$$\mu(P) = \operatorname{tr}(TP) \qquad \text{for all} \quad P \in \mathfrak{P}(\mathbf{Q}^n) \tag{4}$$

where T is a linear operator on $\mathbf{Q}^n$. The operator T is defined by μ uniquely.

Sketch of the Proof. First we consider the case $n = 3$.

Step 0. First we will explain what we will prove. Introduce projections $P_1, P_2, \ldots, P_9$ whose matrices define a linear basis in the nine-dimensional space of all 3×3 rational matrices. We denote $y' \otimes y$ an operator on $\mathbf{Q}^3$ defined by $x \to y'(x)y$. For the sake of definiteness we admit that to a linear basis $e = (1, 0, 0), f = (0, 1, 0), g = (0, 0, 1)$ with the biorthogonal basis $\{e', f', g'\}$ there corresponds the following basis of projections:

$$\begin{array}{ccc} P_1 = e' \otimes e, & P_2 = f' \otimes f, & P_3 = g' \otimes g \\ P_4 = (e' + f') \otimes e, & P_5 = (e' + g') \otimes e, & P_6 = (e' + f') \otimes f \\ P_7 = (f' + g') \otimes f, & P_8 = (e' + g') \otimes g, & P_9 = (f' + g') \otimes g \end{array} \tag{5}$$

Sometimes we will denote by $P_i(e, f, g) = P_i$ the projections defined by a basis $\{e, f, g\}$ in $\mathbf{Q}^3$ in (5).

Suppose that μ satisfies (4); then the matrix T is reproduced by the values $\mu(P_i)$, $i \leq 9$. Therefore, we have only to prove that every signed measure μ on $\mathfrak{P}(\mathbf{Q}^3)$ is defined uniquely by its values on all P_i, $i \leq 9$. In fact, if this is the case, then μ coincides with a signed measure defined by (4) equal to $\mu(P_i)$ for all P_i.

What tool do we have in order to prove the uniqueness? We have just only all the equations

$$\mu(P_1') + \mu(P_2') + \mu(P_3') - \mu(P_1) - \mu(P_2) - \mu(P_3) = 0 \tag{6}$$

for all triples of mutually orthogonal one-dimensional projections $\{P_1', P_2', P_3'\}$. So we start by fixing a "domain of uniqueness" $\{P_i: i \leq 9\}$. Our task is to enlarge this domain to the whole $\mathfrak{P}(\mathbf{Q}^3)$.

Step 1. Let us now consider the set $\mathfrak{P}_0$ of all projections whose matrix elements with respect to the basis $\{e, f, g\}$ take values only in the set $\{-1, 0, +1\}$. Obviously the set $\mathfrak{P}_0$ contains all the basic projections P_i. The cardinality of this set is equal to 87.

Computer Lemma. If two signed measures μ and ν coincide on some nine linearly independent elements of $\mathfrak{P}_0$, then these signed measures coincide on the whole $\mathfrak{P}_0$.

Proof of the lemma consists in the calculation of the range of the system of all equations of type (6) where the triples $\{P_1', P_2', P_3'\}$ are taken from $\mathfrak{P}_0$. The computer says that the number of such equations is equal to 105 and the range of the system is equal to 78 ($= 87 - 9$).

Step 2. The set $\mathfrak{P}_0$ is a large one and contains linear bases of the form $\{P_i(x, y, z): i \leq 9\}$ which are distinct from the basis (P_i) and which are defined as (P_i) by linear bases $\{x, y, z\}$ of $\mathbf{Q}^3$ distinct from $\{e, f, g\}$. By means of this we can enlarge the domain of uniqueness $\mathfrak{P}_0$ of the signed measure and prove the following.

Fundamental Lemma. If a signed measure is defined on the nine elements of the basis $(P_i(x, y, z)$: $i \leq 9)$, then this signed measure is defined uniquely on every projection whose matrix elements with respect to the basis (x, y, z) are integers.

Remark 3. Denote a set of all projections in the lemma by $\mathfrak{P}(x, y, z; \mathbf{Z})$. The following step is more complicated. In order to understand the situation let us consider a simple Gleason signed measure μ defined by (1) and an operator $T = P_1(e, f, g)$ and consider some projection P with a matrix $[p_{ij}]$ such that $p_{11} = 1/n$ where n is a large integer. If we succeed in proving the theorem, then the equality $\mu(P) = 1/n$ should be proved too. We can use in our proof only equations of the type (6), i.e., linear equations with coefficients -1, $+1$, 0 where the right-hand sides are integers [they are the numbers $\mu(P)$ in such equations where $P \in \mathfrak{P}(e, f, g; \mathbf{Z})$]. So the determinant of the system must be divisible by n. Still, to invent a matrix [without any relation to (6)] containing only numbers $\{1, 0, -1\}$ with a determinant dividing n is not easy.

Step 3. We consider now the set $\mathfrak{P}(e, f, g; p)$ of all projections such that all matrix elements with respect to the basis (e, f, g) are numbers of type m/p, where m is an integer and p is a fixed power of a prime number. We present $\mathfrak{P}(e, f, g; p)$ as a union of some sets of type $\mathfrak{P}(x, y, z; \mathbf{Z})$ where a triple $\{x, y, z\}$ runs over some set $\mathfrak{B}$ of bases. In addition, the sets $\mathfrak{P}(x, y, z; \mathbf{Z})$ are such that the nine-element basis $\{P_i(x, y, z)$: $i \leq 9\}$ in $\mathfrak{P}(x, y, z; \mathbf{Z})$ has the following property: only two elements of this basis do not belong to $\mathfrak{P}(e, f, g; \mathbf{Z})$. We connect every such $\mathfrak{P}(x, y, z; \mathbf{Z})$ with another one $\mathfrak{P}(x_1, y_1, z_1; \mathbf{Z})$ so that they have four common elements. These four elements define four equations which connect the basic elements in both sets. The determinant of the system is not null. This includes both sets $\mathfrak{P}(x, y, z; \mathbf{Z})$ and $\mathfrak{P}(x_1, y_1, z_1; \mathbf{Z})$ simultaneously in the domain of uniqueness and so is the whole $\mathfrak{P}(e, f, g; p)$.

Step 4 finishes the proof for $n = 3$. We consider a projection P with matrix elements m/pq, where m is integer and p, q are fixed powers of different prime numbers. We prove that P is included in a triple $\{P, P', P''\}$ where $P' \in \mathfrak{P}(e, f, g; p)$ and $P'' \in \mathfrak{P}(e, f, g; q)$. So a number $\mu(P)$ is defined uniquely by $\mu(P')$ and $\mu(P'')$. Therefore, the general case is reduced to that in step 3.

Step 5 proves by induction the theorem for an arbitrary n.

Remark 4. In fact in steps 1, 2, and 5 we have proved Theorem 6 for $\mathfrak{P}(\mathbf{Z}^n)$.

Theorem 6 admits another pure algebraic formulation:

Theorem 7. Every rational-valued additive function on the set of all idempotents of Hom($\mathbf{Q}^n$, $\mathbf{Q}^n$), where $n \geq 3$, admits an additive extension onto Hom($\mathbf{Q}^n$, $\mathbf{Q}^n$).

Except for finite-dimensional rational linear spaces, Theorem 7 is true for finite-dimensional spaces over residue fields. The proof is the same as that of the fundamental lemma, but the value of the determinant in step 1 is important. This determinant is equal to 1. When we consider a finite field, we represent it as a linear space over a residue field. So we have the following result.

Theorem 8. Let F be a finite field, $n \geq 3$. Every additive function on the set of all idempotents of Hom(F^n, F^n) taking values in F admits an additive extension onto the whole Hom(F^n, F^n).

In concluding the paper, I list several unsolved problems.

Problem A. Can Theorem 3 be generalized to signed measures defined on the OMP of all projections on a real or p-adic Banach space X?

Problem B. Can various spaces that are non-Banach and even not locally convex be taken as H in Theorem 3?

Problem C. A curious case is that when X is the space $L^0(0, 1)$ of all random variables defined on the probability space (0, 1) with the Lebesgue measure, endowed with the topology of convergence in probability. Is the OMP $\mathfrak{P}$ on the space $L^0(0, 1)$ nonatomic?

Problem D. Does at least one pair of closed subspaces that are topological complements of each other exist in any complete metric linear space?

Problem E. Given the condition of Theorem 2, is it possible using only the OMP-structure to restore by L some lattice T_1 such that L is constructed by T_1 as in Theorem 1? It seems that this is impossible. In fact, let T_1 itself be an OMP. Then we can present an element (a, b) in L as a pair (a, b'). Therefore, we cannot be sure that we restore a, but not b'. There is of interest the result of Ovchinnikov (1993), who proved that the family of all subOMPosets $\Pi_A(H)$ is invariant under all automorphisms of the OMP $\mathfrak{P}(H)$.

Problem F. Let R, R_1 be two rings, $\mathfrak{P}$ be the set of all idempotents of R, and μ: $\mathfrak{P} \to R_1$ be an additive operator. When does μ admit an additive extension to the whole R?

I hope that the generalization of Theorem 6 for linear spaces over an arbitrary field of the characteristic 0 is true, too. Such a result would give a possibility to solve Problem A.

Problem G. It would be of interest to prove Theorem 6 for the OMP $\Pi(\mathbf{Q}^n)$ of all projections which are Hermitian with respect to the natural scalar product defined by some basis. This problem seems to be very difficult. Even if we succeed to find some "domain of uniqueness" of a signed measure, it should be difficult to enlarge it, i.e., to make a transition from one basis related to our "domain of uniqueness" to another one. The situation differs from that of the pair $\Pi(H)$, $\mathfrak{P}(H)$; the Π case is more difficult.

It would appear to be of interest to examine such theorems for the lattice of Hermitian projections over residue rings considered by J. Flachsmeyer at this conference. But this Flachsmeyer problem seems to be a difficult problem in number theory.

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REFERENCES

Dorofeev, S. V., and Sherstnev, A. N. (1990). Frame type functions and their applications, *Izvestia Vysshikh Uchebnykh Zavedenii. Seriya Matematika*, **1990**(4), 24–30.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space, *Journal of Mathematics and Mechanics*, **6**(6), 885–894.

Maeda, F., and Maeda, S. (1970). *Theory of Symmetric Lattices*, Springer-Verlag, Berlin.

Matvejchuk, M. S. (1981). The description of finite measures in semifinite algebras, *Functional Analysis and Applications*, **15**(3), 41–53.

Matvejchuk, M. S. (1991). The description of measures on the logics of skew projections. III, *Izvestia Vysshikh Uchebnykh Zavedenii. Seriya Matematika*, **1991**(5), 35–41.

Mushtari, D. H. (1989). Projection logics in Banach spaces, *Izvestia Vysshikh Uchebnykh Zavedenii. Seriya Matematika*, **1989**(8), 59–70.

Mushtari, D. H. (1994). The Gleason theorem for linear spaces over rationals, in *Abstracts of International Conference "Algebra and Analysis" in Honour of N. G. Chebotarev*, Vol. 1, p. 139.

Ovchinnikov, P. G. (1993). Automorphisms of the poset of skew projections, *Journal of Functional Analysis*, **115**(1), 184–189.